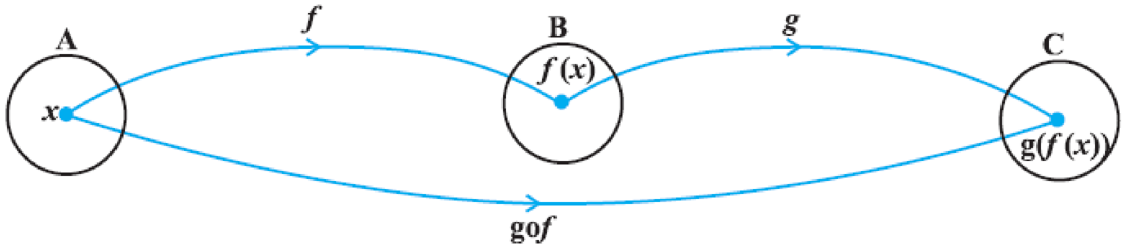
Composition of Functions

Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be two functions. Then the composition of f and g, denoted by gof, is defined as the function gof : A → C given by

$$gof\,(x) = g(f\,(x)),\ \forall\, x \in A.$$



Q. Let $f, g : \boldsymbol{R} \rightarrow \boldsymbol{R}$ be two functions defined as $f\,(x) = |x| + x$ and $g\,(x) = |x| - x$, $\forall\, x \in \boldsymbol{R}$. Then, find $f \circ g$ and $g \circ f$.

Solution.

Here $f(x) = |x| + x$ which can be redefined as

$$f(x) = \begin{cases} 2x \text{ if } x \geq 0 \\ 0 \text{ if } x < 0 \end{cases}$$

Similarly, the function g defined by $g(x) = |x| - x$ may be redefined as

$$g(x) = \begin{cases} 0 \text{ if } x \geq 0 \\ -2x \text{ if } x < 0 \end{cases}$$

Therefore, $g \circ f$ gets defined as :

For $x \geq 0$, $(g \circ f)(x) = g(f(x) = g(2x) = 0$

and for $x < 0$, $(g \circ f)(x) = g(f(x) = g(0) = 0$.

Consequently, we have $(g \circ f)(x) = 0$, $\forall\, x \in \mathbf{R}$.

Vinod A V,
PGT Maths, JNV Thrissur

Similarly, $f \circ g$ gets defined as:

For $x \geq 0$, $(f \circ g)(x) = f(g(x) = f(0) = 0$,

and for $x < 0$, $(f \circ g)(x) = f(g(x)) = f(-2x) = -4x$.

i.e. $$(f \circ g)(x) = \begin{cases} 0, x > 0 \\ -4x, x < 0 \end{cases}$$

Properties of Composition of Functions

- The composition of functions is not commutative.
  $i.e. f \circ g \neq g \circ f$
- The composition of functions is associative.
  $i.e$, if f, g, h are three functions such that $(f \circ g) \circ h$ and $f \circ (g \circ h)$ exist, then $(f \circ g) \circ h = f \circ (g \circ h)$.
- The composition of two bijections is a bijection. $i.e$, if f and g are two bijections, then $g \circ f$ is also a bijection.

  Proof

  Let $f : A \to B$ *and* $g : B \to C$ be two bijections. To prove

  $g \circ f : A \to C$ is also a bijection.

  Suppose $gof\ (x_1) = gof\ (x_2)$

  $\Rightarrow g(f(x_1)) = g(f(x_2))$,

  $\Rightarrow f(x_1) = f(x_2)$, since g is one-one

  $\Rightarrow x_1 = x_2$, since f is one-one

  $\Rightarrow gof$ is one-one.

  Now to prove onto.
  Given an arbitrary element $z \in C$, there exists a pre-image y of z under g such that $g(y) = z$, since g is onto.
  Further, for $y \in$ B, there exists an element x in A with $f(x) = y$, since f is onto.
  Therefore, $gof\ (x) = g(f(x)) = g(y) = z$, showing that gof is onto.

- Let $f : A \to B$. Then, $f \circ I_A = I_B \circ f = f$ $i.e$ the composition of any function with the identity function is the function itself.

Invertible Function

A function $f : X \to Y$ is defined to be invertible, if there exists a function $g : Y \to X$ such that

$g\ o\ f = I_x$ and $f\ o\ g = I_x$.

The function g is called the inverse of f and is denoted byf^{-1}.

Thus, if $f: A \to B$ is a bijection, then $f^{-1}: B \to A$ is such that

$f(x) = y \Leftrightarrow f^{-1}(y) = x$.

Properties of Inverse of a Function

- The inverse of a bijection is unique.
- The inverse of a bijection is also a bijection.
- Let $f : X \to Y$ and $g : Y \to Z$ be two invertible functions. Then $g\ o\ f$ is also invertible with $(g\ o\ f)^{-1} = f^{-1}\ o\ g^{-1}$.

Proof

Since f and g are invertible both functions are bijections, hence $g \circ f$ is also a bijection and so invertible.

Now to show that $(gof)^{-1} = f^{-1}og^{-1}$, it is enough to show that $(f^{-1}og^{-1})o(gof) = I_x$ and $(gof)o(f^{-1}og^{-1}) = I_z$.

Now,

$(f^{-1}og^{-1})o(gof) = ((f^{-1}og^{-1})\ og)\ of$, since composition of functions are associative.

$= (f^{-1}o(g^{-1}og))\ of$,

$= (f^{-1}\ oI_x)\ of$, by definition of g^{-1}

$= I_x$.

Similarly, it can be shown that $(gof\ )o\ (f^{-1}og^{-1}) = I_z$.

Q. Let $f : X \to Y$ be an invertible function. Show that the inverse of f^{-1} is $f, i.e., (f^{-1})^{-1} = f$.

Vinod A V,
PGT Maths, JNV Thrissur