DETERMINANTS

To every square matrix $A = [a_{ij}]$ of order n, we can associate a number (real or complex) called determinant of the matrix A, written as **det A** or $|\boldsymbol{A}|$, where a_{ij} is the $(i,\ j)^{th}$ element of A.

ie. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then determinant of A is written as

$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \det A.$

- **Only square matrices have determinants.**

Determinant of a matrix of order one

Let A = [a] be the matrix of order 1, then determinant of A is defined to be equal to a.

Determinant of a matrix of order two

ie., if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a matrix of order two. Then the determinant of A is defined as

$\det A = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$

***Determinant of a matrix of order* 3**

Determinant of a matrix of order three can be determined by expressing it in terms of second order determinants. This is known as expansion of a determinant along a row (or a column).

There are six ways of expanding a determinant of order 3. Corresponding to each of three rows (R_1, R_2 and R_3) and three columns (C_1, C_2 and C_3) and each way gives the same value.

Consider the determinant of a square matrix $A = [a_{ij}]_{3\times3}$, i.e.,

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Expansion along first Row (R_1)

Step 1

Multiply first element a_{11} of R_1 by $(-1)^{1+1}$ $[(-1)^{\text{sum of suffixes in } a_{11}}]$ and with the second order determinant obtained by deleting the elements of first row (R_1) and first column (C_1) of | A | as a_{11} lies in R_1 and C_1

i.e., $(-1)^{1+1}a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

Step 2

Multiply 2nd element a_{12} of R_1 by $(-1)^{1+2}$ $[(-1)^{\text{sum of suffixes in } a_{12}}]$ and with the second order determinant obtained by deleting the elements of first row (R_1) and first column (C_2) of | A | as a_{12} lies in R_1 and C_2

i.e., $(-1)^{1+2}a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

Step 3

Multiply 3rd element a_{13} of R_1 by $(-1)^{1+3}$ $[(-1)^{\text{sum of suffixes in } a_{13}}]$ and with the second order determinant obtained by deleting the elements of first row (R_1) and first column (C_3) of | A | as a_{13} lies in R_1 and C_3

i.e., $(-1)^{1+3}a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

Step 4

Now the expansion of determinant of A, that is, | A | written as sum of all three terms obtained in steps 1, 2 and 3 above is given by

$$|A| = (-1)^{1+1}a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2}a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{1+3}a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

MINORS AND COFACTORS

Minor of an element

To each element of a square matrix, a number called its minor is associated.

The minor of an element is the value of the determinant obtained by deleting the row and column containing the element.

Minor of an element a_{ij} of a square matrix is the determinant obtained by deleting its i^{th} row and j^{th} column in which element a_{ij} lies.

Minor of an element a_{ij} is denoted by M_{ij}.

Remark Minor of an element of a square matrix of order $n(n \geq 2)$ is a determinant of order $n - 1$.

Cofactor of an element

The cofactor of an element a_{ij} in a square matrix is the minor of a_{ij} multiplied by $(-1)^{i+j}$

It is usually denoted by C_{ij},

Thus,

Cofactor of a_{ij} = C_{ij} = $(-1)^{i+j}M_{ij}$

Note:-

- The sum of the product of elements of any row (or column) with their corresponding cofactors is always equal to its Determinant value

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Now, $a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$

$$(-1)^{1+1}a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2}a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{1+3}a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = |A|$$

- If elements of a row (or column) are multiplied with cofactors of any other row (or column), then their sum is zero.

Properties of Determinants

For any square matrix A, |A| satisfies the following properties.

- $|A'| = |A|$, where $A' = transpose\ of\ matrix\ A$.
- If we interchange any two rows (or columns), then sign of the determinant changes.
- If any two rows or any two columns in a determinant are identical (or proportional), then the value of the determinant is zero.

- Multiplying a determinant by k means multiplying the elements of only one row (or one column) by k.

$$k\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} ka & b & c \\ kd & e & f \\ kg & h & i \end{vmatrix}$$

- If elements of a row (or a column) in a determinant can be expressed as the sum of two or more elements, then the given determinant can be expressed as the sum of two or more determinants.

$$\begin{vmatrix} a+p & b & c \\ d+q & e & f \\ g+r & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} p & b & c \\ q & e & f \\ r & h & i \end{vmatrix}$$

- If to each element of a row (or a column) of a determinant the equi multiples of corresponding elements of other rows (columns) are added, then value of determinant remains same.

i.e., the value of determinant remain same if we apply the operation

$R_i \to R_i + kR_j \; or \; C_i \to C_i + k\,C_j$.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a+kb & b & c \\ d+ke & e & f \\ g+kh & h & i \end{vmatrix}$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a+kb & b & c \\ d+ke & e & f \\ g+kh & h & i \end{vmatrix}$$

$$= \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} kb & b & c \\ ke & e & f \\ kh & h & i \end{vmatrix}$$

$$= \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} + k\begin{vmatrix} b & b & c \\ e & e & f \\ h & h & i \end{vmatrix}$$

$$= \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} + k \times 0$$

$$= \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

- If all the elements of a row (or column) are zeros, then the value of the determinant is zero.
- If all the elements of a determinant above or below the main diagonal consists of zeros, then the value of the determinant is equal to the product of diagonal elements.
- Let A and B are two matrices of same order, then

$$|AB| = |A||B|$$

- If A is square matrix of order n and k be a scalar, then

$$|kA| = k^n|A|$$

Application of Determinants

Area of a Triangle

In earlier classes, we have studied that the area of a triangle whose vertices are $(x_1,\ y_1)$, $(x_2,\ y_2)$ and $(x_3,\ y_3)$, is given by the expression $\frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$.

Now this expression can be written in the form of a determinant as

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Condition of collinearity of three points :

Let $A(x_1,\ y_1)$, $B(x_2,\ y_2)$ and $C(x_3,\ y_3)$ be three points then $A,\ B,\ C$ are collinear if area of $\Delta ABC = 0$

$ie.,\ \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0.$

$\Rightarrow \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0.$

Equation of a line passing through the given two points

Equation of line joining two points $P(x_1,\ y_1)$& $Q(x_2,\ y_2)$ is given by

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

Singular Matrix

A square matrix A is said to be singular if its determinant is zero, ie. $|A| = 0$, otherwise we say that A is non-singular.

ADJOINT OF A SQUARE MATRIX

Adjoint of a given matrix is the transpose of the matrix whose elements are the cofactors of the elements of the given matrix. It is denoted by $Adj\ A$

- Adjoint of a Square matrix of Order 2

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $Adj\ A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

- Adjoint of a Square matrix of Order 2

$$ie,\ if\ A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Then $Adj\ A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$

$ie, Adj\ A = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$.

Theorem:

If A be any given square matrix of order *n*, then

$\boldsymbol{A(Adj\ A)\ =\ (Adj\ A)\ A\ =\ |A|I}$,

where I is the identity matrix of order *n*

INVERSE OF A MATRIX

For a given matrix A, if there exists a matrix B such that

$$\boldsymbol{AB\ =\ BA\ =\ I}$$

then B is called the multiplicative inverse of A.

We write this as $B =\ A^{-1}$.

Properties of Inverse

- We know that $\boldsymbol{A.(Adj\ A)\ =\ (Adj\ A).A\ =\ |A|I}$

$$\Rightarrow A.\frac{Adj\ A}{|A|} = \frac{Adj\ A}{|A|}.A =$$

$I,\qquad where\ I\ is\ the\ identity\ matrix\ of\ order\ n$,

Hence $A^{-1} = \frac{Adj\ A}{|A|}$, provided $|A| \neq 0$

Hence a square matrix A is invertible if and only if A is non-singular.

- Every invertible matrices possess a unique inverse

- If A is invertible, then $(A^{-1})^{-1} = A$
- Let $A,\ B,\ C$ be square matrices of same order, if A is an invertible matrix, then

  $AB = AC \Rightarrow B = C$

  $BA = CA \Rightarrow B = C.$
- Reversal Law: If A & B are invertible matrices of same order, then

$$(AB)^{-1} = B^{-1}A^{-1}.$$

- If A is an invertible matrix , then

$$(A^T)^{-1} = (A^{-1})^T$$

- Let A be a non-singular matrix order n, then

$$|Adj\ A| = |A|^{n-1}$$