Continuity and differentiability

(Q) Show that the function f defined as follows is continuous at $x=2$ but not differentiable at $x=2$

$$f(x) = \begin{cases} 3x-2, & \text{if } 0<x\le 1 \\ 2x^2-x, & \text{if } 1<x\le 2 \\ 5x-4, & \text{if } x>2 \end{cases}$$

Sol

Continuity at $x=2$

$$f(2) = 2(2)^2-2 = 2\times4-2 = 8-2 = 6$$

LHL at $x=2$

$$\lim_{x\to 2^-} f(x) = \lim_{h\to 0} f(2-h) = \lim_{h\to 0} 2(2-h)^2-(2-h)$$

$$= 2(2-0)^2-(2-0)$$

$$= 2(2)^2-2$$

$$= 8-2$$

$$= 6$$

RHL at $x=2$

$$\lim_{x\to 2^+} f(x) = \lim_{h\to 0} f(2+h)$$

$$= \lim_{h\to 0} 5(2+h)-4$$

$$= 5(2+0)-4$$

$$= 10-4$$

$$= 6$$

$\therefore$ LHL = RHL = $f(2) = 6$

$\therefore$ f is continuous at $x=2$.

Differentiability at $x=2$

$$LHD = \lim_{x\to 2^-} \frac{f(x)-f(a)}{x-a}$$

$$= \lim_{h\to 0} \frac{f(2-h)-f(2)}{-h}$$

$$= \lim_{h\to 0} \frac{2(2-h)^2-(2-h)-6}{-h}$$

$$= \lim_{h\to 0} \frac{2(4+h^2-4h)-2+h-6}{-h}$$

$$= \lim_{h\to 0} \frac{8+2h^2-8h-2+h-6}{-h}$$

$$= \lim_{h\to 0} \frac{2h^2-7h}{-h}$$

$$= \lim_{h\to 0} \frac{h(2h-7)}{-h}$$

$$= \frac{2(0-7)}{-1} = -7$$

$\therefore LHD = -7$

$$RHD = \lim_{x\to 2^+} \frac{f(x)-f(2)}{x-2} = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h}$$

$$= \lim_{h\to 0} \frac{5(2+h)-4-6}{h}$$

$$= \lim_{h\to 0} \frac{10+5h-10}{h} = \lim_{h\to 0} \frac{5h}{h} = 5$$

$\therefore RHD = 5$

$\therefore LHD \neq RHD$

$\therefore f$ is not differentiable at $x=2$

Q) Find the value of 'a' and 'b' so that the function f defined as follows is continuous

$$f(x) = \begin{cases} x+2 & \text{if } x \le 2 \\ ax+b & \text{if } 2 < x < 5 \\ 3x-2 & \text{if } x \ge 5 \end{cases}$$

Sol Since f is continuous at everywhere

Continuity at $x=2$

$f(2) = x+2 = 2+2 = 4$; Clearly LHL $= 4$

RHL at $x=2$

$\lim_{x \to 2^+} f(x) = \lim_{h \to 0} f(2+h)$

$= \lim_{h \to 0} a(2+h)+b$

$= a(2+0)+b$

$= 2a+b$

Since f is continuous at $x=2$

$\therefore$ RHL $= f(2)$

$\boxed{2a+b=4} \rightarrow ①$

Continuity at $x=5$

$f(5) = 3x-2 = 3\times5-2 = 15-2 = 13$

RHL $= f(5) = 13$

LHL at $x=5$

$\lim_{x \to 5^-} f(x) = \lim_{h \to 0} f(5-h)$

$= \lim_{h \to 0} a(5-h)+b$

$= a(5-0)+b$

$= 5a+b$

Since f(x) is continuous at $x=5$

$\therefore$ LHL = RHL = f(5)

$\Rightarrow \boxed{5a+b=13} \rightarrow (2)$

from (1) and (2)

$$2a+b=4$$
$$5a+b=13$$
$$(-)\ (-)\ (-)$$

Eq(1) - Eq(2) $\quad -3a = -9$

$$a = \frac{-9}{-3} \Rightarrow$$

$$\boxed{a=3}$$

put $a=3$ in Eq(1) $\quad 2a+b=4$

$$2\times 3+b=4$$
$$6+b=4$$
$$b=4-6=-2$$

$$\therefore \boxed{b=-2}$$

$$\therefore \boxed{a=3,\ b=-2}$$

Q) prove that the function $f(x)=|x-3|+|x-4|$ is not differentiable at $x=3, x=4$

Sol $\quad f(x)=|x-3|+|x-4|$

redefine the function

$$f(x)=\begin{cases} -(x-3)-(x-4), & \text{if } x<3 \\ (x-3)-(x-4), & \text{if } 3\le x<4 \\ (x-3)+(x-4), & \text{if } x\ge 4 \end{cases}$$

$$f(x) = \begin{cases} -x+3-x+4, & \text{if } x < 3 \\ x-3-x+4 & \text{if } 3 \le x < 4 \\ x-3+x-4, & \text{if } x \ge 4 \end{cases}$$

$$f(x) = \begin{cases} 7-2x, & \text{if } x<3 \\ 1 & \text{if } 3 \le x < 4 \\ 2x-7, & \text{if } x \ge 4 \end{cases}$$

<u>Differentiability at $x=3$</u>; $f(3)=1$

$$LHD = \lim_{x \to 3^-} \frac{f(x)-f(3)}{x-3}$$

$$= \lim_{h \to 0} \frac{f(3-h)-f(3)}{-h}$$

$$= \lim_{h \to 0} \frac{7-2(3-h)-1}{-h}$$

$$= \lim_{h \to 0} \frac{7-6+2h-1}{-h}$$

$$= \lim_{h \to 0} \frac{2h}{-h}$$

$$= -2$$

$$\boxed{LHD=-2}$$

$$RHD = \lim_{x \to 3^+} \frac{f(x)-f(3)}{x-3} = \lim_{h \to 0} \frac{f(3+h)-f(3)}{h}$$

$$= \lim_{h \to 0} \frac{1-1}{h} = \lim_{h \to 0} \frac{0}{h} = 0$$

RHD = 0 ∴ f is not differentiable at $x=3$

∴ LHD ≠ RHD

Similarly discuss differentiability of $x = y$

Derivatives

Implicit functions

1) If $\sin y = x \sin(a+y)$, PT $\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$

Sol
$$\sin y = x \sin(a+y)$$
$$x = \frac{\sin y}{\sin(a+y)}$$

diff w.r.t 'y'

$$\frac{dx}{dy} = \frac{\sin(a+y)\cdot\cos y - \sin y \cos(a+y)}{\sin^2(a+y)}$$

$$\frac{dx}{dy} = \frac{\sin((a+y)-y)}{\sin^2(a+y)}$$

$$\frac{dx}{dy} = \frac{\sin a}{\sin^2(a+y)}$$

$$\boxed{\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}}$$

HW 1) If $\cos y = x \cos(a+y)$, PT $\frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$

2) If $\sin y = x \cos(a+y)$, PT $\frac{dy}{dx} = \frac{\cos^2(a+y)}{\cos a}$

21| If $x\sqrt{1+y}+y\sqrt{1+x}=0$, $(x\neq y)$, PT $\frac{dy}{dx}=\frac{-1}{(1+x)^2}$

Sol Given that

$x\sqrt{1+y}+y\sqrt{1+x}=0, \quad x\neq y$

$x\sqrt{1+y}=-y\sqrt{(1+x)}$

$x^2(1+y)=y^2(1+x)$

$x^2+x^2y=y^2+y^2x$

$x^2-y^2+x^2y-y^2x=0$

$(x+y)(x-y)+xy(x-y)=0$

$(x-y)(x+y+xy)=0$

$x+y+xy=0 \quad \text{Since } x\neq y$

$x+y(1+x)=0$

$y(1+x)=-x$

$y=\frac{-x}{1+x}$.

$\frac{dy}{dx}=-\left[\frac{(1+x)\times 1-x(0+1)}{(1+x)^2}\right]$

$=-\left[\frac{1+x-x}{(1+x)^2}\right]$

$\frac{dy}{dx}=-\frac{1}{(1+x)^2}$

Inverse Trig. Function

(Q) Find $\frac{dy}{dx}$, if $y = \tan^{-1}\left(\frac{5x}{1-6x^2}\right)$

Sol $y = \tan^{-1}\left(\frac{3x+2x}{1-3x\times 2x}\right)$

$y = \tan^{-1}(3x) + \tan^{-1}(2x)$

$\frac{dy}{dx} = \frac{1}{1+(3x)^2}\times 3 + \frac{1}{1+(2x)^2}\times 2$

$\frac{dy}{dx} = \frac{3}{1+9x^2} + \frac{2}{1+4x^2}$

(Q) Find $\frac{dy}{dx}$ if $y = \sin^{-1}\left(\frac{2^{x+1}\cdot 3^x}{1+36^x}\right)$

Sol $y = \sin^{-1}\left(\frac{2\cdot 2^x\cdot 3^x}{1+(6^x)^2}\right)$

$y = \sin^{-1}\left(\frac{2\cdot(2\times 3)^x}{1+(6^x)^2}\right)$

$y = \sin^{-1}\left(\frac{2\times 6^x}{1+(6^x)^2}\right)$

$y = 2\tan^{-1}(6^x)$

$\frac{dy}{dx} = 2\times\frac{1}{1+(6^x)^2}\cdot 6^x\log 6$

$= \frac{2\cdot 6^x\cdot\log 6}{1+36^x}$

Q) If $y = \tan^{-1}\left(\dfrac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)$, find $\dfrac{dy}{dx}$

Sol

$$\sqrt{1+\sin x} = \cos x/2 + \sin x/2$$

$$\sqrt{1-\sin x} = \cos x/2 - \sin x/2$$

$$\sqrt{1+\sin x}+\sqrt{1-\sin x} = 2\cos x/2$$

$$\sqrt{1+\sin x}-\sqrt{1-\sin x} = 2\sin x/2$$

$$y = \tan^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)$$

$$y = \tan^{-1}\left(\frac{2\cos(x/2)}{2\sin(x/2)}\right)$$

$$y = \tan^{-1}(\cot(x/2))$$

$$y = \tan^{-1}\left[\tan(\pi/2 - x/2)\right]$$

$$y = \pi/2 - x/2$$

$$\frac{dy}{dx} = 0 - \frac{1}{2}$$

$$\frac{dy}{dx} = -1/2$$

Logarithmic Differentiation

1) $y = x^{\sin x} + (\sin x)^{\cos x}$, find $\frac{dy}{dx}$.

Sol $y = x^{\sin x} + (\sin x)^{\cos x}$

Let $u = x^{\sin x}$; $v = (\sin x)^{\cos x}$

$y = u + v$

$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \rightarrow (1)$

Now $u = x^{\sin x}$

Take logarithm on both sides

$\log u = \log x^{\sin x}$

$\log u = \sin x \cdot \log x$

diff w.r.t x

$\frac{1}{u} \cdot \frac{du}{dx} = \sin x \times \frac{1}{x} + (\log x) \cdot \cos x$

$\frac{du}{dx} = u \left[\frac{\sin x}{x} + (\cos x) \log x \right]$

$$\boxed{\frac{du}{dx} = x^{\sin x} \left(\frac{\sin x}{x} + \cos x (\log x) \right)} \rightarrow (1)$$

Similarly $v = (\sin x)^{\cos x}$

Take logarithm on both sides

$\log v = \log (\sin x)^{\cos x}$

$\log v = \cos x \cdot \log (\sin x)$

diff w.r.t x

$\frac{1}{v} \cdot \frac{dv}{dx} = \cos x \times \frac{1}{\sin x} \cdot \cos x + (\log \sin x) \times -\sin x$

$\frac{dv}{dx} = v \left[\cos x \cdot \cot x - \sin x \log (\sin x) \right]$

$\frac{dv}{dx} = (\sin x)^{\cos x} \left[\cos x \cdot \cot x - \sin x \log (\sin x) \right]$

$\therefore \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} = x^{\sin x} \left[\frac{\sin x}{x} + \cos x (\log x) \right] + (\sin x)^{\cos x} \left[\cos x \cdot \cot x - \sin x \cdot \log (\sin x) \right]$

Q1 If $x^y = e^{x-y}$, Show that $\frac{dy}{dx} = \frac{\log x}{[\log(ex)]^2}$

sol

$$x^y = e^{x-y}$$

Take logarithm on both sides

$$\log x^y = \log e^{x-y}$$

$$y \log x = (x-y) \log e$$

$$y \log x = (x-y) \times 1$$

$$y \log x = x-y$$

$$y + y \log x = x$$

$$y[1+\log x] = x$$

$$y = \frac{x}{[1+\log x]}$$

$$\frac{dy}{dx} = \frac{(1+\log x)\times 1 - x(0+\frac{1}{x})}{(1+\log x)^2}$$

$$= \frac{1+\log x - x \times \frac{1}{x}}{(\log e+\log x)^2}$$

$$= \frac{\cancel{1}+\log x - \cancel{1}}{[\log(ex)]^2}$$

$$\frac{dy}{dx} = \frac{\log x}{[\log(ex)]^2}$$

Parametric Equations – DIFFERENTIATION

1) If $x = a\left[\cos t + \log\left(\tan\frac{t}{2}\right)\right]$, $y = a\sin t$, find the value of $\frac{d^2y}{dx^2}$

Sol:- $x = a\left[\cos t + \log\left(\tan\frac{t}{2}\right)\right]$

$$\frac{dx}{dt} = a\left[-\sin t + \frac{1}{\tan(t/2)} \times \sec^2(t/2) \times \frac{1}{2}\right]$$

$$= a\left[-\sin t + \frac{1}{2} \times \frac{1}{\frac{\sin(t/2)}{\cancel{\cos(t/2)}}} \times \frac{1}{\cos^{\cancel{2}}(t/2)}\right]$$

$$= a\left[-\sin t + \frac{1}{2\sin(t/2)\cos(t/2)}\right]$$

$$= a\left[-\sin t + \frac{1}{\sin 2(t/2)}\right]$$

$$= a\left[-\sin t + \frac{1}{\sin t}\right]$$

$$= a\left[\frac{1-\sin^2 t}{\sin t}\right]$$

$$= a \cdot \frac{\cos^2 t}{\sin t}$$

$$= a \cdot \frac{\cos t}{\sin t} \times \cos t$$

$$\frac{dx}{dt} = a(\cot t)(\cos t)$$

$y = a\sin t$

$$\frac{dy}{dt} = a \cdot \cos t$$

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{\cancel{a}\cancel{\cos t}}{\cancel{a}(\cot t)\cancel{(\cos t)}} = \frac{1}{\cot t} = \tan t$$

$$\boxed{\frac{dy}{dx} = \tan t}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \times \frac{dt}{dx} = \frac{d}{dt}(\tan t) \cdot \frac{dt}{dx}$$

$$= \sec^2 t \times \frac{1}{\left(\frac{dx}{dt}\right)}$$

$$= \sec^2 t \times \frac{1}{a \cdot \cot t \cdot \cos t}$$

$$= \frac{\sec^2 t}{a} \times \tan t \cdot \sec t$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{\sec^3 t \cdot \tan t}{a}}$$

Q) If $x = \sqrt{a^{\sin^{-1} t}}$; $y = \sqrt{a^{\cos^{-1} t}}$, show that $\frac{dy}{dx} = \frac{-y}{x}$

Sol $x = \sqrt{a^{\sin^{-1} t}}$; $y = \sqrt{a^{\cos^{-1} t}}$

$$xy = \sqrt{a^{\sin^{-1} t} \cdot a^{\cos^{-1} t}}$$

$$xy = \sqrt{a^{\sin^{-1} t + \cos^{-1} t}}$$

$$xy = \sqrt{a^{\pi/2}}$$

$$\boxed{xy = a^{\pi/4}}$$

diff w.r.t x

$$x \cdot \frac{dy}{dx} + y \times 1 = 0$$

$$x\frac{dy}{dx} = -y$$

$$\boxed{\frac{dy}{dx} = \frac{-y}{x}}$$

Second order Derivative:-

(1) If $y = A\cdot e^{mx} + Be^{nx}$, then show that

$$\frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny = 0$$

Sol

$$y = Ae^{mx} + Be^{nx} \rightarrow (1)$$

$$\frac{dy}{dx} = A\cdot e^{mx}\cdot m + Be^{nx}n$$

$$\boxed{\frac{dy}{dx} = mAe^{mx} + nBe^{nx}} \rightarrow (2)$$

$$\frac{d^2y}{dx^2} = mAe^{mx}\cdot m + n\cdot Be^{nx}\cdot n$$

$$\boxed{\frac{d^2y}{dx^2} = m^2Ae^{mx} + n^2Be^{nx}} \rightarrow (3)$$

Now $\quad LHS = \frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny$

$$= \left[m^2Ae^{mx} + n^2Be^{nx}\right] - (m+n)(mAe^{mx} + nBe^{nx}) + mn\left[Ae^{mx} + Be^{nx}\right]$$

$$= \cancel{m^2Ae^{mx}} + \cancel{n^2Be^{nx}} - \cancel{m^2Ae^{mx}} - \cancel{mnBe^{nx}} - \cancel{mnAe^{mx}} - \cancel{n^2Be^{nx}} + \cancel{mnAe^{mx}} + \cancel{mnBe^{nx}}$$

$= 0$

$= RHS$

$\therefore LHS = RHS$

2) If $y = \sin^{-1} x$. Show that $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 0$

Proof

$y = \sin^{-1} x \rightarrow (1)$

diff. w.r.t. x

$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$

$\sqrt{1-x^2}\frac{dy}{dx} = 1 \rightarrow (2)$

diff. w.r.t. x

$\sqrt{1-x^2}\frac{d^2y}{dx^2} + \frac{dy}{dx} \times \frac{1}{2\sqrt{1-x^2}} \times (0-2x) = 0$

$\sqrt{1-x^2}\frac{d^2y}{dx^2} - \frac{x}{\sqrt{1-x^2}}\frac{dy}{dx} = 0$

$\boxed{(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 0}$

Q3) If $y = (\sec^{-1} x)^2$, Show that

$x^2(x^2-1)\frac{d^2y}{dx^2} + (2x^3-x)\frac{dy}{dx} = 2$

Proof

$y = (\sec^{-1} x)^2 \rightarrow (1)$

diff. w.r.t. x

$\frac{dy}{dx} = 2\sec^{-1} x \times \left(\frac{-1}{x\sqrt{x^2-1}}\right)$

$\boxed{x.\sqrt{x^2-1}\frac{dy}{dx} = -2\sec^{-1} x} \rightarrow (2)$

diff w.r.t x

$$x\sqrt{x^2-1}\,\frac{d^2y}{dx^2}+(1).\sqrt{x^2-1}\,\frac{dy}{dx}+x.\frac{dy}{dx}\times\frac{1}{2\sqrt{x^2-1}}(2x-0)$$

$$=2.\left(\frac{-1}{x\sqrt{x^2-1}}\right)$$

$$x\sqrt{x^2-1}\,\frac{d^2y}{dx^2}+\sqrt{x^2-1}\,\frac{dy}{dx}+\frac{x^2}{\sqrt{x^2-1}}.\frac{dy}{dx}=\frac{2}{x\sqrt{x^2-1}}$$

multiply both sides by $x\sqrt{x^2-1}$

$$x^2(x^2-1)\frac{d^2y}{dx^2}+x(x^2-1)\frac{dy}{dx}+x^3\frac{dy}{dx}=2$$

$$x^2(x^2-1)\frac{d^2y}{dx^2}+\left[x^3-x+x^3\right]\frac{dy}{dx}=2$$

$$x^2(x^2-1)\frac{d^2y}{dx^2}+(2x^3-x)\frac{dy}{dx}=2$$

Q3) If $y=e^{a\sin^{-1}x}$, $-1\leq x\leq 1$, show that

$$(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}-a^2y=0$$

Sol

$$y=e^{a\sin^{-1}x}\rightarrow(1)$$

$$\frac{dy}{dx}=\left(e^{a\sin^{-1}x}\right)a.\frac{1}{\sqrt{1-x^2}}$$

$$\boxed{\frac{dy}{dx}=\frac{a}{\sqrt{1-x^2}}\,y}\rightarrow(2)$$

$$\sqrt{1-x^2}\,\frac{dy}{dx}=ay\rightarrow(3)$$

diff w.r.t x

$$\sqrt{1-x^2}.\frac{d^2y}{dx^2}+\frac{1}{2\sqrt{1-x^2}}\times(0-2x).\frac{dy}{dx}=a.\frac{dy}{dx}$$

$$\sqrt{1-x^2}\,\frac{d^2y}{dx^2} - \frac{x}{\sqrt{1-x^2}}\frac{dy}{dx} = \frac{a^2y}{\sqrt{1-x^2}}$$

Multiply both sides by $\sqrt{1-x^2}$

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = a^2y$$

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0$$

(4) If $x = \sin\left(\frac{1}{a}\log y\right)$, show that

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0$$

Sol $x = \sin\left(\frac{1}{a}\log y\right)$

$$\sin^{-1}x = \frac{1}{a}\log y$$

$$a\sin^{-1}x = \log y$$

$$y = e^{a\sin^{-1}x}$$

Same as above problem

(see it)

Rolle's Theorem

Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b), such that $f(a)=f(b)$ where 'a' and 'b' are some real numbers. Then there exist some c in (a,b) such that $f'(c)=0$

Q1) verify ~~Rol~~ Rolle's Theorem for
$$f(x)=(x-1)(x-2)^2 \text{ in } [1,2]$$

Sol Given that $f(x)=(x-1)(x-2)^2$ in $[1,2]$

$$a=1,\ b=2$$

Clearly $f(x)$ is a polynomial function

$\therefore f$ is continuous on $[1,2]$

$\therefore f$ is differentiable on $(1,2)$

Now $f(1)=(1-1)(1-2)^2=0\times(-1)^2=0$

$f(2)=(2-1)(2-2)^2=1\times 0^2=0$

$\therefore f(1)=f(2)$

$\therefore f(x)$ satisfies all the conditions of Rolle's Theorem.

Now $f'(x)=(x-1)\cdot 2(x-2)(1-0)+(x-2)^2(1-0)$

$$f'(x)=2(x-1)(x-2)+(x-2)^2$$
$$=(x-2)[2(x-1)+(x-2)]$$
$$=(x-2)[2x-2+x-2]$$
$$f'(x)=(x-2)(3x-4)$$

Now $f'(c)=0$

$$(c-2)(3c-4)=0$$

$c-2=0$ or $3c-4=0$

$c=2$, (or) $3c=4$

$c=4/3$

Clearly $c=4/3 \in (1,2)$

$\therefore$ There exists some $c=4/3 \in (1,3)$

such that $f'(c)=0$

$\therefore$ Rolle's Theorem verified.

(Q) Verify Rolle's theorem for

$f(x)=\cos x+\sin x$ on $[0,2\pi]$

Sol $f(x)=\cos x+\sin x$; $a=0$, $b=2\pi$

clearly f is continuous and differentiable

Now $f(0)=\cos 0+\sin 0=1+0=1$

$f(2\pi)=\cos 2\pi+\sin 2\pi=1+0=1$

$\therefore f(0)=f(2\pi)$

$\therefore$ f satisfies all the conditions of Rolle's thm

Now $f'(x)=-\sin x+\cos x$

Let $f'(c)=0$

$-\sin c+\cos c=0$

$\sin c=\cos c$

$\tan c=1$

$c=\pi/4$, $3\pi/4$

$\therefore$ There exists $\pi/4 \in (0,2\pi)$

and $\frac{3\pi}{4} \in [0,2\pi]$ such that $f'(c)=0$

$\therefore$ Rolle's Theorem verified.

Mean Value Theorm / Lagrange's Mean Value Theorm

Let $f: [a,b] \to \mathbb{R}$ be a continuous function on $[a,b]$ and differentiable on (a,b). Then there exists some c in (a,b) such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Q) Verify Mean Value Theorm (Lagrange's)

$f(x) = x^2 - 3x + 2$ in $[-1,2]$ $\quad a=-1,\ b=2$

Sol: Given function $f(x) = x^2 - 3x + 2$

Since f is polynomial function

$\therefore$ f is continuous on $[-1,2]$

and f is differentiable on $[-1,2]$

Now $f(-1) = (-1)^2 - 3(-1) + 2 = 1+3+2 = 6$

$f(2) = 2^2 - 3(2) + 2 = 4 - 6 + 2 = 0$

$f'(x) = 2x - 3$

Now $f'(c) = \dfrac{f(2)-f(-1)}{2-(-1)}$

$$2c - 3 = \frac{6-0}{3} = \frac{6}{3}$$

$2c - 3 = 2$

$2c = 2+3$

$2c = 5 \qquad c = 5/2 = 2.5$

$\therefore$ clearly $2.5 \in (-1,2)$

$\therefore$ there exists $2.5 \in (-1,2)$ such that $f'(c) = \dfrac{f(b)-f(a)}{b-a}$

$\therefore$ Mean value Theorm verified

Differentiation of a function w.r.t to another function

Let f(x) and g(x) are two functions then

$$\frac{df}{dg} = \frac{\left(\frac{df}{dx}\right)}{\frac{dg}{dx}}$$

Ex: Differentiate f(x) with respect to g(x)

1) $f(x) = \log(1+x^2)$; $g(x) = \tan^{-1}x$

$$\frac{df}{dx} = \frac{1}{1+x^2} \times (0+2x) \qquad \frac{dg}{dx} = \frac{1}{1+x^2}$$

$$\frac{df}{dx} = \frac{2x}{1+x^2} ; \qquad \frac{dg}{dx} = \frac{1}{1+x^2}$$

$$\therefore \frac{df}{dg} = \frac{\left(df/dx\right)}{\left(dg/dx\right)} = \frac{\left(\frac{2x}{1+x^2}\right)}{\left(\frac{1}{1+x^2}\right)} = \frac{2x}{\cancel{(1+x^2)}} \times \frac{\cancel{(1+x^2)}}{1}$$

$$= 2x$$

Derivative of Infinite series

1) If $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \ldots \infty}}}}$

prove that $\frac{dy}{dx} = \frac{\cos x}{2y-1}$

Sol $y = \sqrt{\sin x + y}$

$y^2 = \sin x + y$

$y^2 - y = \sin x$

diff w.r.t x

$$2y\frac{dy}{dx} - \frac{dy}{dx} = \cos x$$

$$(2y-1)\frac{dy}{dx} = \cos x$$

$$\frac{dy}{dx} = \frac{\cos x}{2y-1}$$