LEARNING OUTCOMES OF DIFFERENTIAL EQUATIONS

- Students observe the differential equations and identifies the order and degree of the differential equations.
- Checking the given function is a solution of differential euation or not.
- Forming the differential equation in different cases
- General and Particular solutions of a differential equation
- Different methods of solving first order and first degree differential equations.

  --- By Variable –seperable Method

  --- Solving Homogeneous Differential Equations

  --- Solving Linear Differential Equations

REAL LIFE APPLICATIONS

- As you know that derivatives are nothing but rate of change of one quantity compared to other quantity.
- So we see many circumstances in the real life where one quantity changes with respect to other quantity.
- We use this type of derivatives in different fields like in Population growth, Culture of Bacteria(calculation of Corona virus growth), Weather and Climate prediction, Traffic flow, financial markets, water pollution, chemical reactions, suspension bridges , brain function, rockets, tumor growth, radioactive decay, airflow across aeroplane wings, planetary motion, electrical circuits, vibrations in strings and many more

- Differential equations enter into the different fields Physics, Chemistry, Biology, Astronomy, Music, Economics, Medicine, Aeronautics etc.,
- So, in order to study them we have to learn the fundamental study of derivatives.

- Observe some of the Equations

$$y = x^2 - 2x$$

$$x^2 y^2 + y = 1$$

$$y^2 - 2xy\frac{dy}{dx} = 0$$

$$\frac{d^2y}{dx^2} + y = 0$$

$$xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0$$

Some are only functions of x and y alone.

Some of those equations contains derivatives

Next

DIFFERENTIAL EQUATIONS

An equation containing derivatives of the dependant variable with respect to the independent variable is called Differential equation.

- DEFINITION: A differential equation involving derivatives of the dependent variable with respect to only one independent variable is called an Ordinary differential equation.

$$2\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0 \quad \leftarrow$$

derivative form

$$y''+y'-2y=3$$

This is a differential equation because it has 'derivative' components in it

differential form

$$\frac{d^2y}{dx^2}+\frac{dy}{dx}-2y=3$$

This is a differential equation because it has 'differential' components in it

$$y^2+y^{-1}-2y=3$$

This is NOT a differential equation because it does not have 'differential' nor 'derivative' components in it

$$y''+y'-2y$$

This is NOT a differential equation because it is not a form of equation (no 'equal' sign) even though it has 'derivative' component in it

Go

NOTATIONS FOR DERIVATIVES

- The following notations are used for derivatives

$$\frac{dy}{dx} = y' \qquad \frac{d^2y}{dx^2} = y'' \quad and \quad \frac{d^3y}{dx^3} = y'''$$

For the derivatives of the higher order it is inconvenient to use so many dashes as supersuffix , so we use

$$the\ notation\ y_n\ for\ n\ th\ order\ derivative\ \frac{d^ny}{dx^n}$$

ORDER OF A DIFFERENTIAL EQUATION

- It is defined as the order of the highest derivative of dependant variable with respect to the independant variable involved in the given differential equation.

[Find the order]

DEGREE OF A DIFFERENTIAL EQUATION

- The highest power(positive integral index) of the highest order derivative involved in the given differential equation.

Find the degree

Find the Order and Degree

NOTE: ORDER AND DEGREE OF A DIFFERENTIAL EQUATION BE NEVER NEGATIVE

SOLUTION OF D.E

- Generally the solutions of the type

$$x^2 - 5 = 0$$

$$x^2 - 5x + 6 = 0 \; or \; \sin x - \cos x = 1$$

are real or complex numbers that satisfies the equations.

Where as the solution of Differential equation is A FUNCTION which satisfies the given D.E like

$$y = e^x + 1 \; is \; the \; solution \; of \; y'' - y' = 0$$ also

$$y = e^x + C \; is \; the \; solution \; of \; y'' - y' = 0$$

GENERAL AND PARTICULAR SOLUTIONS OF D.E

$y = e^x + 1 \; is \; the \; solution \; of \; y'' - y' = 0$ also

$y = e^x + C \; is \; the \; solution \; of \; y'' - y' = 0$

The solution which contains the arbitrary constants is called the **general solution**

where as the solution free from arbitrary constants is the **particular solution**

Formation of D.E whose general solution is given

- Form the D.E representing the family of curves y=mx where m is arbitrary constant.
- Differentiating both sides of the equation

y=mx

w.r.t x we get

$$\frac{dy}{dx} = m$$

Substituting the value of m in the given equation, we get

$$y = \frac{dy}{dx}x \text{ or } x\frac{dy}{dx} - y = o$$

- Q. Form the differential equations representing the family of circles touching the x-axis at the origin.
- Q. Form the differential equations representing the family of ellipses having foci on x-axis and centre at the origin

- NOTE: The order of differential equation representing the family of curves is same as the number of arbitrary constants present in the equation corresponding to the family of curves.

METHODS OF SOLVING FIRST ORDER AND FIRST DEGREE DIFFERENTIAL EQUATIONS

1. Variable-Seperable method
2. Homogeneous Method
3. Equations of the form Linear Differential Equation

VARIABLE-SEPERABLE METHOD

- Equations of the form $\frac{dy}{dx} = f(x)$

In this method , we write $dy = f(x)dx$

Then integrate both sides to get the solution.

- Equations of the form $f(x)dx = g(y)dy$

This type also can be solved by integrating both sides i.e $\int f(x)dx = \int g(y)dy + c$

VARIABLE-SEPERABLE METHOD

Find the general and particular solution of the differential equation given that y=1 when x=0

$$\frac{dy}{dx} = -4xy^2$$

Solu:

$$If\ y \neq 0, the\ given\ D.E\ can\ be\ written\ as$$

$$\frac{dy}{y^2} = -4x\,dx \longrightarrow 1$$

Integrating both sides of equation 1 , we get

$$\int \frac{dy}{y^2} = -4\int x\,dx$$

$$\frac{-1}{y} = -2x^2 + C$$

$$y = \frac{1}{2x^2 - C}$$

. *Solve:* $(x+2)\dfrac{dy}{dx} = x^2 + 4x - 9,\ x \neq -2.$

The given equation can be written as $\dfrac{dy}{dx} = \dfrac{x^2+4x-9}{(x+2)}$

$$\Rightarrow \quad dy = \frac{x^2+4x-9}{(x+2)}dx\,.$$

Now integrating both sides, we get

$$y = \int \frac{x^2+4x-9}{(x+2)}dx = \int \left(x + 2 - \frac{13}{x+2} \right) dx$$

$\Rightarrow \quad y = \dfrac{x^2}{2} + 2x - 13 \log |x+2| + C$ is the required solution.

HOMOGENEOUS FUNCTION

- A function F(x,y) is a homogeneous function of degree n if

$$F(x,y) = x^n g\left(\frac{y}{x}\right) \; or \; y^n h\left(\frac{x}{y}\right)$$

Example: $F(x,y) = 2x - 3y = x^1(2 - \frac{3y}{x})$

$$F(x,y) = \frac{x+2y}{x-y} = \frac{x\left(1+2\frac{y}{x}\right)}{x\left(1-\frac{y}{x}\right)}$$

(or)

- A function F(x,y) is said to be homogeneous function of degree n if

$$F(\lambda x, \lambda y) = \lambda^n F(x,y) \; for\ any\ non\ zero\ constant\ \lambda$$

Consider the examples

$F(x,y) = 2x - 3y$ then

$$F(\lambda x, \lambda y) = \lambda(2x - 3y) = \lambda F(x,y)$$

HOMOGENEOUS D E

- A differential equation of the form

$$\frac{dy}{dx} = F(x,y) \text{ is said to be homogeneous}$$

$$\text{if } F(x,y) \text{ is a homogeneous function}$$

Example: $x\frac{dy}{dx} = y(\log y - \log x + 1)$

$$\frac{dy}{dx} = \frac{y}{x}\left(\log\frac{y}{x} + 1\right)$$

SOLVING H D E

- If the homogeneous differential equation is in the form

$$\frac{dy}{dx} = F(x, y) where\ F(x, y) is\ homogeneous\ of\ degree\ zero$$

$$then\ we\ make\ substitution\ \frac{y}{x} = v\ i.e$$

$$y = xv\ and\ we\ proceed\ further\ to\ find$$

$$the\ general\ solutio\ \ by\ writing$$

$$\frac{dy}{dx} = F(x, y) = h\left(\frac{y}{x}\right)$$

Solve the differential equation $(1 + e^{2x})\,dy + (1+ y^2)e^x dx = 0$ *given that when* $x = 0, y = 1$.

The given equation can be written as $\dfrac{dy}{1+y^2} = -\dfrac{e^x}{1+e^{2x}}dx$.

Integrating both sides, we get

$$\int\frac{dy}{1+y^2} = -\int\frac{e^x}{1+e^{2x}}dx \Rightarrow \int\frac{dy}{1+y^2} = -\int\frac{1}{1+t^2}dt \text{ where } t = e^x \Rightarrow dt = e^x dx.$$

$$\Rightarrow \quad \tan^{-1}y = -\tan^{-1}(t) + C \Rightarrow \tan^{-1}y = -\tan^{-1}(e^x) + C. \qquad \text{...(1)}$$

Now, put $x = 0, y = 1$ in (1) $\Rightarrow \dfrac{\pi}{4} = -\dfrac{\pi}{4} + C \Rightarrow C = \dfrac{\pi}{2}$.

$\Rightarrow$ (1) becomes $\tan^{-1}y = \dfrac{\pi}{2} - \tan^{-1}(e^x) \Rightarrow \tan^{-1}y = \cot^{-1}(e^x)$

$$\Rightarrow \quad \tan^{-1}y = \tan^{-1}\left(\frac{1}{e^x}\right) \Rightarrow y = \frac{1}{e^x}.$$

Hence, $y = \dfrac{1}{e^x}$ is the required solution.

Q. Show that the family of curves for which the slope of the tangent at any point (x,y) on it is $\frac{x^2+y^2}{2xy}$ is given by $x^2 - y^2 = cx$

Solution We know that the slope of the tangent at any point on a curve is $\frac{dy}{dx}$.

Therefore,

$$\frac{dy}{dx} = \frac{x^2+y^2}{2xy}$$

$$\frac{dy}{dx} = \frac{1+\frac{y^2}{x^2}}{\frac{2y}{x}}$$

To solve consider the substitutuion,

$$\frac{y}{x} = v \ \ then\ y = vx\ by\ differentiating\ \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = \frac{1 + v^2}{2v}$$

$$\frac{2v}{1 - v^2}dv = \frac{dx}{x}$$

$$\frac{2v}{v^2 = 1}dv = -\frac{dx}{x}$$

$$\int \frac{2v}{v^2 - 1}dv = -\int \frac{1}{x}dx$$

$$\log[v^2 - 1] = -\log[x] + \log C_1 \qquad \log[(v^2 - 1)](x) = \log[C_1] \qquad (v^2 - 1) = \pm C_1$$

Replacing v by $\frac{y}{x}$ *we get*

$$\left(\frac{y^2}{x^2} - 1\right)x = \pm C_1$$

$$(y^2 - x^2) = \pm C_1\, x \text{ or } x^2 - y^2 = Cx$$

THE END

LINEAR DIFFERENTIAL EQUATION

- A DIFFERENTIAL EQUATION OF THE FORM $$\frac{dy}{dx} + Py = Q$$ where P and Q are constants or functions of x only is known as a first order linear differential equation.

Example:

$$\frac{dy}{dx} + \frac{y}{x} = x^2$$

$$\frac{dy}{dx} + y \sec x = \tan x$$

ALGORITHM FOR SOLING L.D.E

- 1. Write the differential equatin in the form of $\frac{dy}{dx} + Py = Q$ and obtain P, Q.

2. Find the integrating factor I.F = $e^{\int P\,dx}$

3. Find the solution of the differential equation using $y\,(I.F) = \int Q\,.\,(I.F)\,dx + c$

FINDING WHETHER THE GIVEN DE IS LINEAR OR NOT

- A differential equation is linear, if the dependent variable and the derivative appear in the equation is of first degree.
- Example: $\frac{dy}{dx} + xy = x^3$

$$\frac{dy}{dx} + 2y\cot x = 3x^2 \csc^2 x$$

Solve: $\dfrac{dy}{dx} + y\tan x = \cos^3 x.$

The given equation is of the form $\dfrac{dy}{dx} + Py = Q$, where $P = \tan x$ and $Q = \cos^3 x$.

$\therefore$ I.F. $= e^{\int P dx} = e^{\int \tan x dx} = e^{\log(\sec x)} = \sec x$.

$\therefore$ The solution is $y(\text{IF}) = \int Q\,(\text{I.F.})\,dx + C$

$$\Rightarrow \quad y(\sec x) = \int \cos^3 x . \sec x dx + C$$

$$= \int \cos^2 x dx + C = \int \left(\frac{1 + \cos 2x}{2}\right) dx + C$$

$$= \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right] + C$$

Hence, the required solution is $y(\sec x) = \dfrac{1}{2}\left[x + \dfrac{\sin 2x}{2}\right] + C$.

Solve: $\frac{dy}{dx} + y \sec x = \tan x$.

The given equation is of the form $\frac{dy}{dx} + Py = Q$ where $P = \sec x$ and $Q = \tan x$.

$$\text{I.F.} = e^{\int P dx} = e^{\int \sec x dx} = e^{\log(\sec x + \tan x)} = \sec x + \tan x$$

$\therefore$ The solution is $y(\text{I.F.}) = \int Q\,(\text{I.F.})\,dx + C$

$$\Rightarrow \quad y(\sec x + \tan x) = \int \tan x(\sec x + \tan x)dx + C$$

$$\Rightarrow \quad y(\sec x + \tan x) = \int (\tan^2 x + \sec x \tan x)dx + C$$

$$\Rightarrow \quad y(\sec x + \tan x) = \int (\sec^2 x - 1 + \sec x \tan x)dx + C$$

$$\Rightarrow \quad y(\sec x + \tan x) = \tan x - x + \sec x + C.$$

Hence, $y(\sec x + \tan x) = \tan x + \sec x - x + C$ is the required solution.

THANK YOU